

Data Analysis of Continuous Gravitational Wave Signal: Fourier Transform

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We present the Fourier Transform of continuous gravitational wave for arbitrary location of detector and source and for any duration of observation time in which both rotational motion of earth about its spin axis and orbital motion around sun has been taken into account. We also give the method to account the spin down of continuous gravitational wave.

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The detection of gravitational waves (GW) from their possible sources, at the present stage, has to largely depend upon the careful study of the expected characteristic of potential sources and the waveforms. In this context, the continuous sources, in particular pulsars, are now receiving much attention [1,2]. These may be the one of the sources to look for the signals in data streams of the long arm laser interferometer and are also attractive sources for currently operating bar detectors. The theoretical research clearly shows that the data analysis will involve processing of very long time series and exploration of a very large parameter space. It turns out that optimal processing of month long data streams is computationally prohibitive [1] and methods to reduce the computation burden are being worked out [4,5].

At the root cause of the excessive computational requirement lies the fourier transform (FT) analysis of the data. In order to detect the signal from the dominant noise one has to analyse the long time observation data ranging from months to year. The output of detector will be Doppler modulated; both frequency and amplitude due to the rotations of earth. The maximum Doppler shift for one kHz signal is $\simeq 21.42 \times 10^{-2}$ [8]. The output depends in a complex manner on the location of the source and the detector. Consequently, the FT is obtained using numerical techniques; one commonly in use is fast fourier transform (FFT). The evaluation of FT analytically is an important step in the data analysis and in contrast to FFT has following advantages:

- The computational time is extremely less and particularly in the case of long data train gains significance.
- We can achieve the resolution of the transform as we need, hence dominant peaks of doppler modulated signals can be accumulated, and
- To make templates for matched filtering in colored noise

In the literature very little attention has been paid to analytical analysis for fourier transform [6-8]. In this letter we address to this problem in its complete generality.

The response $R(t)$ of the ground based laser interferometric detector at time t is a linear combination of the two polarizations h_+ and h_\times of the signal given via

$$R(t) = F_+(t)h_+(t) + F_\times(t)h_\times(t) \quad (1)$$

where F_+ , F_\times are the beam pattern functions and represent the amplitude modulation (AM) part. In Solar System Barycentre (SSB) frame the beam pattern as function of the direction of the incoming wave (θ, ϕ, ψ) , the orientation of the detector angles (α, β, γ) and ϵ , the angle between the equator and the ecliptic may be expressed as [6]

$$F_+ = \frac{1}{2} [2(L^2 - M^2)ZU + (N^2 - P^2)A - (Q^2 - R^2)B] + (LN - MP)C + (LQ + MR)D + (NQ + PR)E, \quad (2)$$

$$F_\times = 2LMZU + NPA - \frac{1}{2}B \sin^2 \theta \sin 2\phi + (LP + MN)C + (MQ - LR)D + (PQ - NR)E \quad (3)$$

where

$$\begin{aligned} A &= 2XY \cos^2 \epsilon - \sin^2 \epsilon \sin^2 \alpha \sin 2\gamma + \sin 2\epsilon (X \sin \alpha \sin \gamma - Y \sin \alpha \cos \gamma), \\ B &= 2XY \sin^2 \epsilon - \cos^2 \epsilon \sin^2 \alpha \sin 2\gamma - \sin 2\epsilon (X \sin \alpha \sin \gamma - Y \sin \alpha \cos \gamma), \\ C &= \cos \epsilon (YV + XU) + \sin \epsilon (V \sin \alpha \sin \gamma - U \sin \alpha \cos \gamma), \\ D &= -\sin \epsilon (YV + XU) + \cos \epsilon (V \sin \alpha \sin \gamma - U \sin \alpha \cos \gamma), \\ E &= -2XY \cos \epsilon \sin \epsilon - \cos \epsilon \sin \epsilon \sin^2 \alpha \sin 2\gamma + \cos 2\epsilon (X \sin \alpha \sin \gamma - Y \sin \alpha \cos \gamma), \\ L &= \cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi, \\ M &= \cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi, \\ N &= -\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi, \\ P &= -\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi, \\ Q &= \sin \theta \sin \phi, \quad R = \sin \theta \cos \phi, \\ U &= -\cos \alpha \cos \beta \sin \gamma - \sin \beta \cos \gamma, \\ V &= \cos \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma, \end{aligned}$$

$$\begin{aligned}
X &= \cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma, \\
Y &= -\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma, \\
\beta &= \beta_o + w_r t
\end{aligned}$$

Here β_o represent the initial azimuth of the detector. We have assumed that the arms of the laser interferometer are mutually orthogonal. However, if the angle between the arms of the detector are different than 90° the expression for F_+ and F_\times will simply get multiplied by sine of the angle between the arms of the detector respectively. Inclusion of this parameter does not pose any difficulty as it simply affect the norm of the FT by the same factor.

After algebraic manipulation eq. (2) and (3) may be expressed as

$$F_+ = F_{1+} \cos 2\beta + F_{2+} \sin 2\beta + F_{3+} \cos \beta + F_{4+} \sin \beta + F_{5+} \quad (4)$$

$$F_\times = F_{1\times} \cos 2\beta + F_{2\times} \sin 2\beta + F_{3\times} \cos \beta + F_{4\times} \sin \beta + F_{5\times} \quad (5)$$

F_{i+} and $F_{i\times}$ ($i = 1, 2, 3, 4, 5$) are time independent and are given by

$$\begin{aligned}
F_{1+} &= -2G \cos \alpha \cos 2\gamma + \frac{H \sin 2\gamma}{2} (\cos^2 \alpha + 1), \\
F_{2+} &= H \cos \alpha \cos 2\gamma + G \sin 2\gamma (\cos^2 \alpha + 1), \\
F_{3+} &= I \sin \alpha \cos 2\gamma + J \sin 2\alpha \sin 2\gamma, \\
F_{4+} &= 2J \sin \alpha \cos 2\gamma - \frac{I}{2} \sin 2\alpha \sin 2\gamma, \\
F_{5+} &= \frac{3 \sin^2 \alpha \sin 2\gamma}{2} [H + L^2 - M^2], \\
G &= \frac{1}{2} [(LQ + MR) \sin \epsilon - (LN - MP) \cos \epsilon], \\
H &= \frac{1}{2} [(N^2 - P^2) \cos^2 \epsilon - (L^2 - M^2) + (Q^2 - R^2) \sin^2 \epsilon - (NQ + PR) \sin 2\epsilon], \\
I &= \frac{1}{2} [(Q^2 - R^2) \sin 2\epsilon - (N^2 - P^2) \sin 2\epsilon - 2(NQ + PR) \cos 2\epsilon], \\
J &= \frac{1}{2} [(LN - MP) \sin \epsilon + (LQ + MR) \cos \epsilon]
\end{aligned}$$

and $F_{i\times}$ is related to F_{i+} via

$$F_{i\times}(\theta, \phi, \psi, \alpha, \beta, \gamma, \epsilon) = F_{i+}(\theta, \phi - \frac{\pi}{4}, \psi, \alpha, \beta, \gamma, \epsilon) \quad (6)$$

$$i = 1, 2, 3, 4, 5$$

The two polarisation states of the signal may be taken as

$$h_+(t) = h_{o+} \cos[\Phi(t)], \quad h_\times(t) = h_{o\times} \sin[\Phi(t)]$$

where h_{o+} , $h_{o\times}$ are the time independent amplitude of $h_+(t)$, and $h_\times(t)$. Here $\Phi(t)$ is phase of the GW signal and may be expressed in SSB frame as [7]

$$\begin{aligned}
\Phi(t) &= 2\pi f_o \left[t + \frac{R_{se}}{c} \sin \theta \cos \phi' + \frac{R_e}{c} \sin \alpha \{ \sin \theta (\sin \beta \cos \epsilon \sin \phi + \cos \phi \cos \beta) + \right. \\
&\quad \left. \sin \beta \sin \epsilon \cos \theta \} - \frac{R_{se}}{c} \sin \theta \cos \phi_o + \frac{R_e}{c} \sin \alpha \{ \sin \theta (\sin \beta_o \cos \epsilon \sin \phi + \cos \phi \cos \beta_o) + \right. \\
&\quad \left. \sin \beta_o \sin \epsilon \cos \theta \} \right] \quad (7)
\end{aligned}$$

where, $\phi' = w_{orb}t - \phi$, R_{se} is the distance from the centre of the SSB frame to the centre of the earth, R_e is the radius of the earth, w_{orb} is the angular velocity of the earth around the sun, w_r is the angular velocity of the earth about its spin axis and c is the velocity of light. Here f_o is the frequency of the continuous signal.

The complete response of the detector is given via

$$\tilde{R}(f) = \tilde{R}_+(f) + \tilde{R}_\times(f), \quad (8)$$

$$\tilde{R}_+(f) = \int h_+(t) F_+(t) e^{-i2\pi f t} dt, \quad (9)$$

$$\tilde{R}_\times(f) = \int h_\times(t) F_\times(t) e^{-i2\pi f t} dt \quad (10)$$

Due to the symmetries involved in F_+ and F_\times as given by [eqn. (6)] it is sufficient to evaluate either of the FTs (9) and (10) and the other may be obtained in a simple manner. We will, at present, consider $\tilde{R}_+(f)$ and get

$$\begin{aligned}
\tilde{R}_+(f) &= h_{o+} \left[e^{-i2\beta_o} (F_{1+} + iF_{2+}) \tilde{h}_+(f + 2f_r)/2 + \right. \\
&\quad e^{i2\beta_o} (F_{1+} - iF_{2+}) \tilde{h}_+(f - 2f_r)/2 + \\
&\quad e^{-i\beta_o} (F_{3+} + iF_{4+}) \tilde{h}_+(f + f_r)/2 + \\
&\quad \left. e^{i\beta_o} (F_{3+} - iF_{4+}) \tilde{h}_+(f - f_r)/2 + F_{5+} \tilde{h}(f) \right] \quad (11)
\end{aligned}$$

where

$$\tilde{h}_+(f) = \int h_{o+} \cos[\Phi(t)] \quad (12)$$

and represent the FT arising due to frequency modulation. This means that the AM results in the four side bands at frequencies $f \pm f_r$, $f \pm 2f_r$ around the central frequency f . For the moment, we keep the data interval of the observation time completely arbitrary. We evaluate the FT

$$\tilde{h}(f) = \tilde{h}_+(f) = \int_{n\Delta t}^{(n+1)\Delta t} h_{o+} \cos[\Phi(t)] \quad (13)$$

where n is an integer and Δt is time interval. It is important to point out that it is not easy to evaluate the FT analytically and it is the stage where one has to take recourse to numerical methods. The FT [eq. (13)] after lengthy but straight forward calculation is obtained as a double series

$$\tilde{h}(f) = \frac{h_{o+}\nu}{2w_r} \sum_{k=-\infty}^{k=\infty} \sum_{m=-\infty}^{m=\infty} e^{i\mathcal{A}} \mathcal{B}(\mathcal{Z}, \mathcal{N}) [\mathcal{C}(\lambda, \zeta, \tau) - i\mathcal{D}(\lambda, \zeta, \tau)] \quad (14)$$

where

$$\begin{aligned} \nu &= (f_0 - f)/f_r, \quad (f_r = w_r/2\pi), \\ \mathcal{A} &= \frac{(k+m)\pi}{2} - \mathcal{R}, \\ \mathcal{B}(\mathcal{Z}, \mathcal{N}) &= \frac{J_k(\mathcal{Z})J_m(\mathcal{N})}{\nu^2 - (ak+m)^2}, \\ \mathcal{C}(\lambda, \zeta, \tau) &= \sin(\nu\tau) \cos(ak\tau + m\tau - k\lambda - m\zeta) \\ &\quad - \frac{ak+m}{\nu} \{ \cos(\nu\tau) \sin(ak\tau + m\tau - k\lambda - m\zeta) \\ &\quad + \sin(k\lambda + m\zeta) \}, \\ \mathcal{D}(\lambda, \zeta, \tau) &= \cos(\nu\tau) \cos(ak\tau + m\tau - k\lambda - m\zeta) \\ &\quad + \frac{ak+m}{\nu} \sin(\nu\tau) \sin(ak\tau + m\tau - k\lambda - m\zeta) \\ &\quad - \cos(k\lambda + m\zeta), \\ \mathcal{R} &= \mathcal{Z} \cos \phi + \mathcal{Q} + 2\pi f_o n \Delta t, \\ \mathcal{Z} &= (2\pi/c) f_o R_{se} \sin \theta, \quad \mathcal{N} = \sqrt{\mathcal{P}^2 + \mathcal{Q}^2}, \\ \lambda &= \phi - an\tau, \quad \zeta = \delta - n\tau, \\ \delta &= \tan^{-1} \frac{\mathcal{P}}{\mathcal{Q}}, \quad \tau = w_r \Delta t, \quad a = 1/365.25, \\ \mathcal{P} &= (2\pi/c) f_o R_e \sin \alpha [\cos \beta_o (\sin \theta \cos \epsilon \sin \phi + \cos \theta \sin \epsilon) - \sin \beta_o \sin \theta \cos \phi], \\ \mathcal{Q} &= (2\pi/c) f_o R_e \sin \alpha [\sin \beta_o (\sin \theta \cos \epsilon \sin \phi + \cos \theta \sin \epsilon) + \cos \beta_o \sin \theta \cos \phi], \end{aligned}$$

$J_k(\mathcal{Z})$, $J_m(\mathcal{N})$ represent the bessel function of first kind of integer order k and m respectively.

A case of particular interest arises where one needs to evaluate the FT of the data for T_{obs} time i.e. the limits in (14) are taken as 0 to T_{obs} . The result in this case is obtained by taking

$$n = 0, \quad \Delta t = T_{obs} \implies \tau = w_r T_{obs}$$

Thus we obtain

$$\tilde{h}(f) = \frac{h_{o+}\nu}{2w_r} \sum_{k=-\infty}^{k=\infty} \sum_{m=-\infty}^{m=\infty} e^{i\mathcal{A}} \mathcal{B}(\mathcal{Z}, \mathcal{N}) [\mathcal{C}(\lambda, \zeta, w_r T_{obs}) - i\mathcal{D}(\lambda, \zeta, w_r T_{obs})] \quad (15)$$

Suppose one is interested in evaluating the FT for a day or year observation data then he/she can obtain the FT by putting $n = 0$ in eq. (14), and

$$T_{obs} = \text{one year} \implies \tau = 2\pi/a$$

$$T_{obs} = \text{one day} \implies \tau = 2\pi$$

The result obtained [eq. (14)] is useful in getting the FT of the complete response of a detector of GW being emitted by a pulsar whose frequency is either slowing down or picking up. To account for this aspect, we consider the evaluation in different window of time by splitting the interval $(0, T_{obs})$ in N equal parts, each of interval Δt ($T_{obs} = N \Delta t$) such that the signal over a window may be treated as monochromatic. The strategy is to evaluate the FT over the window and finally to add the result. This process has also been suggested by Brady & Creighton [6] and Schutz [4] in numerical computing and called by them as stacking. The transform [eq. (14)] can be computed easily for data of any observation time. Consequently, the obtained transform may be one of the efficient tool for the detection of continuous gravitational wave. To obtain the result one has to evaluate $\tilde{R}(f)$ in each window by accounting for changing value of f_o .

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